LECTURE 31 APPLIED OPTIMIZATION AND NEWTON'S METHOD

Applied Optimization

(1) Read the problem statements very carefully.
(2) Identify the known quantity and relationship (known as the constraints).
(3) Introduce all variables. If applicable, draw a figure and label all variables. Write down their possible range of values.
(4) Identify what you need to optimize (in physical terms, such as area, volume, total cost, etc.). Sometimes this is given clearly. Sometimes, you need to interpret.
(5) Express what you need to optimize in the simplest manner possible, which may involve multiple variables.
(6) Shrink the number of variables by using the known quantity and relationship.
(7) Do calculus to find **absolute optima**.

Example. Design a one-liter (1000 cubic centimeters) can shaped like a right circular cylinder. What dimensions will use the least material?

Solution. Volume of the cylinder can be computed when we know the radius of the base circle r and the height of the cylinder h. In this problem, they are related by

$$\pi r^2 h = 1000.$$

At the same time, the material one uses pertains to the total surface area of the cylinder, that is,

$$S = 2\pi rh + 2\pi r^2.$$

The volume equation is fixed, and thus known as the constraint of the optimization problem. Constraints usually give us some information about how the variables are related. Looking at the surface area equation (which we want to minimize), we found independent variables r and h. It is always easier to deal with one variable than two. Luckily, the volume equation gives us the dependence between r and h, i.e.

$$h = \frac{1000}{\pi r^2}.$$

Then, we can express

$$S = S(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$$

where we now find the minimum of this function.

We first locate the critical points.

$$S'(r) = -\frac{2000}{r^2} + 4\pi r = 0 \implies r = \sqrt[3]{\frac{500}{\pi}}.$$

To check if it is a local minimum, we either do 1st derivative test or 2nd. We opt for the 2nd derivative test,

$$S''(r) = \frac{4000}{r^3} + 4\pi > 0$$

for all $r > 0$. Therefore, $r = \sqrt[3]{\frac{500}{\pi}}$ is a local minimizer, and in fact, global minimizer (since there are no more critical points).

Example. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution. First, the area of a (unconstrained) rectangle is given by

$$A = 2xy$$

where x is half of length and y is width/height. Now, we need this point (x, y) to live on a circle of radius 2, that is,

$$x^2 + y^2 = 4 \implies y = \sqrt{4 - x^2}.$$

This relationship reduces the area formula to only depend on one variable,

$$A(x) = 2x\sqrt{4 - x^2}.$$

We now simply maximize $A(x)$ on $[0, 2]$ (why $[0, 2]$?)

$$A'(x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} = 0 \implies x = \pm\sqrt{2}.$$

We evaluate $x = \sqrt{2}$ (reject the other because it is not in our domain) and the endpoints,

$$A(0) = 0 = A(2)$$

and

$$A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4 - 2} = 4.$$

This implies that the maximum area is 4 when the rectangle is $\sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long.

Example. An island is 2 miles due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 miles west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 miles and swims at a rate of 3 miles. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution. We want to minimize total time T. This time consists of running time T_{run} and swimming time T_{swim} via

$$T = T_{run} + T_{swim}.$$

Now, suppose we run x miles on the shore and swim y miles in the water. Then we have

$$T = \frac{x}{8} + \frac{y}{3}.$$

All we need now is a relationship between x and y so that this problem becomes one-dimensional. Note that wherever you hit the water, you are horizontal $6 - x$ miles from the island and vertically 2 miles. The hypotenuse of this right triangle is the distance you swim. Therefore,

$$y^2 = (6 - x)^2 + 2^2 \implies y = \sqrt{(6 - x)^2 + 4}.$$

Plugging this into total time, we have

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}, \quad x \in [0, 6].$$

We find critical points which satisfy

$$0 = T'(x) = \frac{1}{8} - \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}} \implies 3\sqrt{(6 - x)^2 + 4} = 8(6 - x)$$

$$\implies 9(6 - x)^2 + 36 = 64(6 - x)^2 \implies 55(6 - x)^2 = 36$$

$$\implies (6 - x)^2 = \frac{36}{55} \implies x = 6 \pm \frac{6}{\sqrt{55}}$$

We reject $x = 6 + \frac{6}{\sqrt{55}}$ since it's outside out domain. Now, we evaluate the endpoints.

$$T(0) = \frac{\sqrt{40}}{3} = \frac{4}{3}\sqrt{10} = \frac{16\sqrt{10}}{12}$$

$$T(6) = \frac{3}{4} + \frac{2}{3} = \frac{3}{4} + \frac{8}{12} = \frac{17}{12}$$

$$T\left(6 - \frac{6}{\sqrt{55}}\right) = \frac{3}{4} - \frac{3}{4\sqrt{55}} + \frac{16}{3\sqrt{55}} = \frac{3}{4} + \frac{1}{\sqrt{55}}\left(\frac{16}{3} - \frac{3}{4}\right) = \frac{3}{4} + \frac{\sqrt{55}}{12}$$

Clearly, $T(0) > T(6) > T\left(6 - \frac{6}{\sqrt{55}}\right)$, which means $T\left(6 - \frac{6}{\sqrt{55}}\right)$ is the absolute minimum.

Newton's Method

Finding zeros of an equation is particularly important, exemplified in the previous section on applied optimization, where we are required to find the critical points of a function, i.e. the zeros of the derivative.

We also have some experience in finding zeros of particular types of functions. For quadratic polynomials, we have the quadratic formula to find the roots, or at least the discriminant to determine whether there is a root or not. For cubic polynomials, we also have a discriminant, but it gets complicated. For higher order polynomials or even nonlinear functions, we don't always have an easy formula for their zeros. A numerical approach must be implemented to approximate the location of the zero.

Newton's method relies on the fact that the function can be locally approximated by its tangent line/linearisation. At a point $x = x_0$,

$$f(x) \approx L(x) = f(x_0) + f'(x_0)(x - x_0).$$

Now, finding the zero x_1 of a line is extremely straightforward, namely,

$$0 = L(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) \implies x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

given that $f'(x_0) \neq 0$. Cool! We just found the zero x_1 of the linear approximation of $f(x)$ at x_0. Since it is an approximation, the zero of $L(x)$ and $f(x)$ are likely not the same. But now, we are going to apply the same method by finding the zero x_2 of the linearisation of $f(x)$ at x_1, that is, we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

given that $f'(x_1) \neq 0$. This means, at the n^{th} step, we have something like

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_{n+1} is the root of the linearisation of $f(x)$ at $x = x_n$. At each stage, we check the absolute difference between x_{n+1} and x_n. If it is not so big, say, below a tolerance level, we then have found the zero of $f(x)$.

Magic? Why does the sequential error $|x_{n+1} - x_n|$ become small? Is it always the case? Convergence proof will require knowledge from Calculus II. Stay tuned!

Let's make sure we know how to use the **iterative formula** as above.

Example. Suppose we want to find the root of $f(x) = x^2 - 2$ on $[0, 2]$. Now, this has a known solution $x = \sqrt{2}$. Can we showcase how good Newton's method is? Newton's method always starts with an initial guess, whose goodness may be guaranteed by the intermediate value theorem, which you apply to narrow down the search space. Suppose, we set

$$x_0 = 1.$$

We find that $f'(x) = 2x$. So our formula becomes,

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

We find

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} = \frac{3}{2}.$$

Then, we continue,

$$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \approx 1.41667$$

and lastly

$$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} \approx 1.41422.$$

The true answer $\sqrt{2} \approx 1.41421$. So within three steps of iteration, we are accurate up to 5 digits!

Remark. Remember the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The function is almost always given, and the initial guess x_0 is sometimes given (if not, then you give your own). Therefore, all you have to do is to find $f'(x)$, simplify the formula if possible, and then evaluate many times. You will be asked to compute several iterations of the method.